An Analysis of Commuting Patterns in Sydney in 2011, using a new Spatial Interaction Model*

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Abstract: Following the seminal work by Wilson (1971), Doubly Constrained Spatial Interaction (DCSI) models have been used to analyse population flows in the form of both commuting (O'Kelly and Lee, 2005, O'Kelly, and Niedzielski, 2008) and also migration. In recent work O'Kelly et al (2012) and O'Kelly (2012) have made conceptual and empirical advances in the application of these models to commuting data. Modelling the pattern of commuting yields both measures of emissivity and accessibility via the (constraint) multipliers and also yields implicit rents and wages by location.

In this exploratory paper, the more sophisticated DCSI model developed by O'Kelly (2012) is estimated via a computational search algorithm in Matlab to calibrate parameter estimates using Travel Zone based journey to work data from the 2011 ABS Census for the Sydney Greater Metropolitan Area. We use the implicit rents to examine whether Alonso's rent gradient can be utilised to identify areas of polycentricity. Also spatial patterns of accessibility are analysed, using Exploratory Spatial Data Analysis. Finally we show that Fotheringham (1983) was correct in highlighting the likelihood of distance decay parameters across different destinations exhibiting global spatial dependence, but there is no evidence of global spatial dependence of origin based distance decay parameters.

1. Introduction

Even though daily urban commuting represents one of many trips undertaken by households, researchers continue to focus on this form of travel in their studies of the relationship between urban form and travel patterns, mainly due the greater availability of journey to work (JTW) data which is typically Census based. Also, by focusing on commuting which is associated with the highest levels of congestion, the maximum demands on the transport system can be identified (Horner, 2004), although commuting flows are no longer synchronised due to hours of work increasingly being non-standard. Both the calculation of current usage and the forecasting of future peak load are complex, but important in the planning of provision of new transport infrastructure.

Following the seminal work by Wilson (1971), researchers have modelled origin destination flows through the adoption of constrained and unconstrained specifications of the spatial interaction (gravity) model. Gravity models are premised on the interaction volume being a function of three variables, namely nodal propulsiveness, nodal attractiveness and the cost associated with the spatial separation of nodes, which is frequently measured by an impedance function based on distance (Fotheringham, 1983; Griffith and Jones, 1980).

Doubly Constrained Spatial Interaction (DCSI) models have been utilised across a range of discipline areas, including economics, regional science, geography and urban studies. These models are often applied to population flows, in particular commuting (O'Kelly and Lee, 2005, O'Kelly, and Niedzielski, 2008, O'Kelly et al, 2012).

In contrast to optimising models of commuting behaviour, which can yield extreme solutions that are sensitive to the chosen urban area (see, for example, White, 1988; Horner, 2002), DCSI models estimate actual commuting flows subject to constraints associated with the overall average commute and total flows from each origin and to each destination.

Writers, including Griffith (1976), Sheppard (1984) and Fotheringham (1983), have been critical of the specification of the gravity model for failing to take account of the relative location or *spatial structure*. Fotheringham (1983) notes that previous studies have usually shown a marked spatial pattern when origin based distance decay parameters are mapped.

O'Kelly et al (2012) and O'Kelly (2012) have made conceptual and empirical advances in the development, manipulation and interpretation of these spatial interaction models. The multipliers associated with the two sets of flow constraints which are identified as measures of emissivity and attraction by location (Wilson, 1974) can be differentiated with respect to the impedance parameter to derive measures of rent and wages. This means that an analysis of Alonso's wage and rent gradients is not reliant on the collection of market data at a high degree of spatial disaggregation. O'Kelly et al (2012) pose the question as to whether these spatial interaction models can signal the benefits of relatively accessible locations. Further, are destination specific attraction factors linked to the spatial distribution of employment centres? Thus the results may provide a novel way of identifying urban polycentricity.

Further, in an application to airline travel, O'Kelly (2012) has shown that the standard spatial interaction model can be refined to accommodate the constraints imposed by the actual origin and destination based average trip lengths. This model generates origin and destination specific decay parameters within an integrated framework, as opposed to piecemeal calculations of decay parameters by origin (Fotheringham, 1983). This gravity model with four sets of constraints again yields measures of emissivity and accessibility and implicit rents and wages by location. We argue that this model yields a richer set of quantitative insights about commuting flows.

In this exploratory paper, we apply this model with the additional constraints to the analysis of commuting flows within the Sydney Greater Metropolitan Area (GMA) based on the Census Journey to Work (JTW) data for 2011. We draw on flow data based on Travel Zones (TZs) defined under the new statistical geography (Bureau of Transport Statistics, 2013, p.12).

In the next section we develop our modelling framework, which is based on the recent methodological developments in DCSI models, and analyse the extant empirical literature. We then outline the data and discuss the algorithm employed to generate the results, which are presented and assessed in Section 4. Concluding comments complete the paper.

2. Literature Review

DCSI Model

The Doubly Constrained Spatial Interaction (DCSI) model yields an entropy maximising estimate of the flows, based on a small number of properties of the actual commuting flows, namely the overall average commute and the observed row and column flow sums of the JTW matrix. The model is underpinned by an impedance function based on imputed travel costs between all locations i and j in the urban area which typically take the form of the Euclidean travel distance, the shortest distance by major road or corresponding travel time. We concur with Charron (2007) that Euclidean distance is the appropriate metric to use. Usually the impedance specification takes the form of a negative exponential function of distance.

The DCSI specification can be written as

$$\text{Max } H = -\sum_i \sum_j T_{ij} \ln T_{ij} \tag{1}$$

$$\begin{aligned} \text{subject to } & \sum_j T_{ij} = O_i \\ & \sum_i T_{ij} = D_j \\ & \sum_i \sum_j T_{ij} C_{ij} / \bar{T} = \bar{C} \end{aligned} \tag{2}$$

where T=[T_{ij}] (i = 1,2,...n; j = 1,2,...m) denotes the JTW matrix; T_{ij} is the number of employees who are resident in location i and work in location j; O_i, D_j are, respectively, the total number of employees resident in location i and the total number of employees who work in location j; and $\bar{T}$ is the total number of employees. C_{ij} is the Euclidean distance between areas i and j and $\bar{C}$ is the overall average trip length. Entropy maximisation is associated with finding the outcome which is the most likely, subject to the three sets of constraints, with H representing an approximation for the objective function (Wilson, 1971; Killen, 1983).

From the first order conditions:

$$T_{ij} = \exp(\lambda_i + \mu_j - \beta C_{ij}) \quad (3)$$

where λ_i, μ_j and β are the unknown Lagrange multipliers corresponding to the Origin (row) and Destination (column) constraints and overall average trip length constraints, respectively.
Equation (3) can then be written in DCSI form as:

$$T_{ij} = A_i O_i B_j D_j \exp(-\beta C_{ij}) \quad (4)$$

where A_i, B_j are respectively the row i and column j balancing factors and $\exp(-\beta C_{ij})$ is the exponential cost function associated with travel between i and j. $\lambda_i = \ln(A_i O_i)$ and $\mu_j = \ln(B_j D_j)$ (Wilson, 1974, quoted in O'Kelly and Niedzielski, 2008).

In a further innovation in the modelling of commuting behaviour, O'Kelly (2012) develops a more sophisticated spatial interaction model via the incorporation of distance decay parameters for each row and column, (β_i, γ_j) rather than the single distance decay parameter, β, in order to impose the requirement that the solution satisfies the average origin based and destination based commuting distances. The model now takes the form:

$$\begin{aligned} T_{ij} &= \exp(\lambda_i + \mu_j - (\varphi + \beta_i + \gamma_j) C_{ij}) \\ &= \exp(\lambda_i) \exp(\mu_j) \exp(-(\varphi + \beta_i + \gamma_j) C_{ij}) \\ &= A_i O_i B_j D_j \exp(-(\varphi + \beta_i + \gamma_j) C_{ij}) \end{aligned} \quad (5)$$

where φ is a shift parameter which increases the overall negative exponential on each distance commuted, thereby marginally reducing the calculated trip lengths, as compared to the actual average trip lengths.

Summing (5) in turn across i and j yields the following expressions for the balancing factors:

$$A_i = 1/\sum_j B_j D_j \exp(-(\varphi + \beta_i + \gamma_j) C_{ij}) \quad (6)$$

and

$$B_j = 1/\sum_i A_i O_i \exp(-(\varphi + \beta_i + \gamma_j) C_{ij}) \quad (7)$$

In order to derive these measures, it is necessary to add another condition. One zonal parameter (the numeraire) is set arbitrarily at unity and added to the first order conditions. The relative magnitudes of the derived measures are not sensitive to the choice of numeraire (Watts, 2012). This requirement for a numeraire also holds for the standard DCSI model. For the numeraire origin, say i, $A_i O_i$ is equal to unity, which implies that the multiplier, λ_i, is equal to 0.

Fotheringham (1983) notes that previous studies of commuting flows have often revealed a marked spatial pattern when origin specific distance decay parameters are mapped. Less accessible locations were associated with larger distance decay parameters. He argues that this spatial structure effect must be in part a consequence of the configuration of destinations around each origin. The model outlined above enables the criticism made by Fotheringham concerning the spatial variation of the distance decay parameters across different origins (and destinations) to be rigorously checked within an integrated framework. In addition to the origin and destination based total flows and the overall average commute, data on the origin and destination based average commute are required. In his empirical application O'Kelly (2012) draws on airline data, which overcomes these data problems.

Solution Properties

The solution has a number of desirable properties, which are reported in a series of papers authored or co-authored by O'Kelly (O'Kelly and Niedzielski, 2008; O'Kelly et al, 2012; O'Kelly, 2012). First, the multiplier for origin i can be written as

Following O'Kelly et al (2012), (5) can be reconfigured and summed over j to yield

$$O_i \exp(-\lambda_i) = \sum_j \exp\left(\mu_j - (\varphi + \beta_i + \gamma_j) C_{ij}\right) \quad (8)$$

Equation (8) is now differentiated with respect to β which after further manipulation can be written as

$$\frac{d\lambda_i}{d\varphi} = \left(\frac{1}{O_i}\right) \sum_j T_{ij} \left(C_{ij} - \frac{d\mu_j}{d\varphi}\right) = \bar{C}_{i.} - \sum_j T_{ij} \frac{d\mu_j}{d\varphi} / \sum_j T_{ij} \quad (9)$$

where $\overline{C_{i.}}$ denotes the mean distance commuted from Origin i. The second term on the right hand side is the weighted sum of the derivatives of the other multipliers, μ_j with respect to ϕ.

Similarly, after summation over i, (5) can be written as:

$$D_j \exp(-\mu_j) = \sum_i \exp\left(\lambda_i - (\varphi + \beta_i + \gamma_j) C_{ij}\right) \quad (10)$$

which, when differentiated with respect to β, yields after manipulation,

$$\frac{d\mu_j}{d\varphi} = \left(\frac{1}{D_j}\right) \sum_i T_{ij} \left(C_{ij} - \frac{d\lambda_i}{d\varphi}\right) = \bar{C}_{.j} - \sum_i T_{ij} \frac{d\mu_j}{d\varphi} / \sum_i T_{ij} \quad (11)$$

where $\overline{C_{.j}}$ denotes the mean distance commuted to Destination j. In the Appendix it is shown that solutions for these derivatives can be readily obtained by matrix inversion.

The origin based derivative with respect to the uniform impedance parameter β, $d\lambda_i/d\phi$, represents the implicit rent and should be non-positive. Hence an appropriate strategy is to choose the numeraire, such that all origin based derivatives are non-positive. Given the invariance of the relative magnitudes of the derivatives (Watts, 2012), this requires that the numeraire is chosen which corresponds to the location which yields the maximum of the origin based derivatives, for any arbitrary choice of numeraire. By imposing this numeraire, its implicit rent will be zero and necessarily the remaining rents will assume a negative sign. At the same time the values of $d\mu_j/d\phi$ (the implicit wage in the destination area) will be positive which accords with convention. O'Kelly et al (2012) exploit equations (9) and (11) to examine the magnitudes of the derivatives in the empirical application of their modelling to Ireland. However, due to the need to impose a numeraire, the relative values of these derivatives assume importance, rather than the absolute magnitudes.

Emissivity, Accessibility and Polycentricity

The ratio of the estimated flows of commuters from origins i,k to a common destination, j is

$$T_{ij}/T_{kj} = A_i O_i B_j D_j \exp\left(-(\varphi + \beta_i + \gamma_j) C_{ij}\right) / \left[A_k O_k B_j D_j \exp\left(-(\varphi + \beta_k + \gamma_j) C_{kj}\right)\right] \quad (12)$$

which, if origin i constitutes the numeraire, can be rewritten as

$$A_k O_k = \left(\frac{T_{kj}}{T_{ij}}\right) / \left[\left(\exp\left(-(\varphi + \beta_i + \gamma_j) C_{ij}\right) / \exp\left(-(\varphi + \beta_k + \gamma_j) C_{kj}\right)\right]\right. \quad (13)$$

Then, if the left hand side exceeds unity so that $\log(A_k O_k) = \lambda_k > 0$, the relative flow to destination j from origin k, as compared to origin i, is greater than is implied by the distance ratio. Thus, based on the estimated flows, origin k has relatively greater emissivity than origin i, irrespective of which destination is being considered. Conversely a negative value of the origin k multiplier implies a lower relative emissivity than is implied by the distance ratio.

Thus the concept of emissivity requires a nuanced understanding, because it is not simply based on the volume of commuting flows from a particular origin, but rather the extent to which these flows are pushed out over a longer relative distance (see O'Kelly et al., 2012 for a similar analysis based on the simple DCSI model).

Likewise, the concept of accessibility does not merely reflect the attraction of commuting flows to a particular location but also the distance over which the attraction takes place. Maintaining an origin based numeraire, say i, the expression corresponding to (13) based on flows from a common origin m to two different destinations, j,k can be written:

$$B_j D_j / B_k D_k = (\frac{T_{mj}}{T_{mk}}) / [(\exp(-(\varphi + \beta_m + \gamma_j) C_{mj}) / \exp(-(\varphi + \beta_m + \gamma_k) C_{mk}] \qquad (14)$$

Then a ratio on the LHS exceeding unity implies that destination j is relatively more accessible, than destination k, irrespective of the choice of origin. A non-CBD destination which is highly accessible may constitute part of a sub-centre, which would signify polycentricity.

In their conceptual paper, Burger and Meijers (2012) refer to morphological polycentricity, namely the 'balance in the size distribution or absolute importance of centres' (p.1130), in which the emphasis is on stocks of employment or population. In addition, they define the functional dimension of polycentricity with its emphasis on linkages between centres via flows would appear to be synonymous with accessibility. They argue that nodality should underpin the measurement of morphological polycentricity, whereas centrality should be the basis of the measurement of functional polycentricity.

They define centrality of a centre *j*, CE_j, in a closed system of cities as

$$CEj = Dj - Lj = \sum_i T_{ij} - T_{jj} \qquad (15)$$

where D_j is the centre's internal and incoming external flows—i.e. its nodality; and L_j represents the internal flows, i.e. those residents of the centre who work there too, so centrality measures the incoming commuting flows (Burger and Meijers, 2012, p.1131). Thus the two distinct properties of their concept of polycentricity sum to the total level of employment at location j. The distribution of employment between residents living in location j and those living outside will influence the measure of accessibility, but the nature of that influence is unclear.

Three types of empirical analysis based on the spatial distribution of employment have been used to identify sub-centres. McMillen (2001) and Redfearn (2007) adopt locally (geographically) weighted regression (GWR) to locate sub-centres in US urban areas (see also McMillen, 2003). Issues have been raised concerning the statistical properties of GWR. Second, Giuliano and Small (1991) employ an arbitrary quantitative technique, in which areas making up each sub-centre must satisfy minimum density and total employment constraints, which ignores the functional dimension of polycentricity. Third, Exploratory Spatial Data Analysis (ESDA), which is mainly used in European studies, identifies areas characterised by a significant spatial association with respect to employment density, namely High-High or High-Low as measured by LISA statistics (Baumont et al, 2004, Guillian et al, 2004, 2006 and Riguelle et al, 2007). While the LISA statistic and also the Getis-Ord Gi* are both local measures of spatial association, they are directly affected by global parameters, namely the mean and variance of employment density for the whole urban area. Thus in a region dominated by a CBD, this is a significant chance that, in contrast to other approaches, sub-centres outside the CBD will not be identified using local spatial statistics.

A second approach to the identification of polycentricity is the examination of the bid rent function, as defined by Alonso (1964). Due to the presence of higher transportation costs, both direct and indirect, further from the CBD, it is argued that residents are prepared to live further away and commute to the CBD if compensated by lower rent. This model assumes that i) the city is located in a featureless plain, with no geographic obstacles such as hills and rivers, which would impact on commuting costs; ii) commuting costs are a linear function of the distance travelled to the CBD; and iii) most jobs are in the CBD and the remainder are distributed uniformly across the remainder of the city. The presence of employment sub-centres (i.e polycentricity) means that the rents are likely to increase close to sub-centres and then continue their decline with increasing distance from the CBD.

3. Data and Methodology

Satisfactory results from Spatial Interaction Models can only be obtained if the region under consideration is largely self-contained, so there are minimal external flows into or out of the region (O'Kelly et al, 2012). For this reason the Greater Sydney Metropolitan Area (GMA) was chosen based on the new 2011 statistical geography, ASGS (Australian Statistical Geography Standard). The closure rates were respectively 99.57 (resident), 99.05 (employment) for the 2941 Travel Zones. There were no working residents in 264 of these but all TZs had local employment. The corresponding rows of the JTW matrix were removed, so it was no longer square.

The solution algorithm was coded in Matlab by the author. The β_i (row) and γ_j (column) exponent vectors were initially set at half the inverse of their respective origin and destination based average commutes. This is in line with the recommendation of Williams (1976, p.93) for the impedance parameter, β_t in the simple DCSI model. The row balancing factor A was initially set as a vector of ones. Then an iterative procedure was used to recursively solve for these column and row balancing vectors, until they satisfied the row and column flow constraints within a specified magnitude.

The β and γ exponent vectors were then adjusted via a generalisation of the calibration technique of Hyman (1969), namely

$$\beta_{it} = \beta_{it-1} * \frac{c^O_{it-1}}{\bar{c}^O_i} \qquad i = 1,2,\ldots n \qquad (16)$$

$$\gamma_{jt} = \gamma_{jt-1} * \frac{c^D_{jt-1}}{\bar{c}^D_j} \qquad j = 1,2,\ldots n \qquad (17)$$

where c^O_i and c^D_j denote the estimated average distance commuted from origin i and commuted to destination j, respectively, and the barred variables represent their actual values. The algorithm continued to iterate until the β and γ vectors stabilised, with the appropriate balancing factors being obtained after each adjustment of the exponent vectors.

This model did not converge for eleven origins and one destination based TZs in 2011. These anomalous results, while relatively insignificant given the number of Travel Zones, were subject to careful scrutiny. Most of these TZs were associated with low total (row or column) flows, but the deletion of the low flow rows and columns was rejected as being arbitrary. The corresponding impedance exponent for the 12 TZs were exceedingly small, which suggested that the Hyman adjustment described above was too restrictive. Accordingly an extra condition was imposed that if the absolute value of the exponent was less than 0.000001, it was multiplied by -1 and, if now negative, the multiplicative factors in (16) and (17), respectively were inverted to generate convergence.

4. Results

Introduction

The Matlab algorithm converged to an error of 9.81 across the total commuting flows of the 2675 origins and 2941 destinations. The total error with respect to the average distance commuted across these TZs was 18.80 with the maximum error for an origin TZ of 1.78 and a destination TZ of 0.40. Five origin TZs and one destination TZ yielded a negative exponential, whereas the origin TZ yielding the maximum error (1.78) had a β exponent of 0.000013. This suggests that a weaker condition for the exponents to change sign was required.

We chose the origin numeraire to ensure that all rents were non-positive, which in turn leads to all implicit wages being non-negative. Our primary focus in analysing the results was to investigate the spatial pattern of implicit rents ($d\lambda/d\phi$) and the measure of accessibility for destinations (μ). The motivation for our focus is to tentatively assess whether the application of this Spatial Interaction Model yields any insights about the emergence of polycentric patterns of employment, with particular reference to the functional interpretation of polycentricity.

Rent Gradient

We estimated the gradient via OLS with the 2675 non-zero values of the implicit rent specified as a linear function of their distance from the centroid of the Inner Sydney SA4 in 2011, which we took to be TZ84, the Strand Arcade. The gradient was significant and negative (-0.62) and the estimated equation had an adjusted R^2 of 0.92. A 99% confidence interval was constructed for the estimated linear function but 1729 observations exceeded the upper bound of the confidence interval. The inclusion of a quadratic distance term to capture the possible non-linearity of the rent gradient led to a reduction of the number of these observations to 1034. A plot of the rent gradient is shown in Figure 1 with the 99% confidence interval based on the non-linear specification.

We checked the spatial weight matrices based on contiguity, but many TZs had no neighbours so we constructed a weight matrix based on the four nearest neighbours. We now turn to consideration of the spatial pattern of employment accessibility (μ). Notwithstanding some reservations about the interpretation of LISAs, we calculated LISAs for the μs, applied 9999 permutations and used significance at the 1% level. 97 TZs exhibited HH with respect to the accessibility measure of which just 46 were found in the metropolitan Sydney area, which we define as consisting of the 14 Sydney Statistical Areas 4 (SA4s) under the new statistical geography. The limited concentrations of accessibility within the Sydney metropolitan area, is in contrast to studies, such as Pfister et al. (2000) and Watts (2008), which adopt the more traditional techniques for identifying polycentricity. Further analysis is warranted. Table 1 shows the SA4 locations of HH combinations of the accessibility variable and the associated number of TZs and Figure 2 maps the HH and LL combinations of accessibility.

Figure 1 Implicit Rent and Distance from the Sydney CBD: 99% Confidence Interval

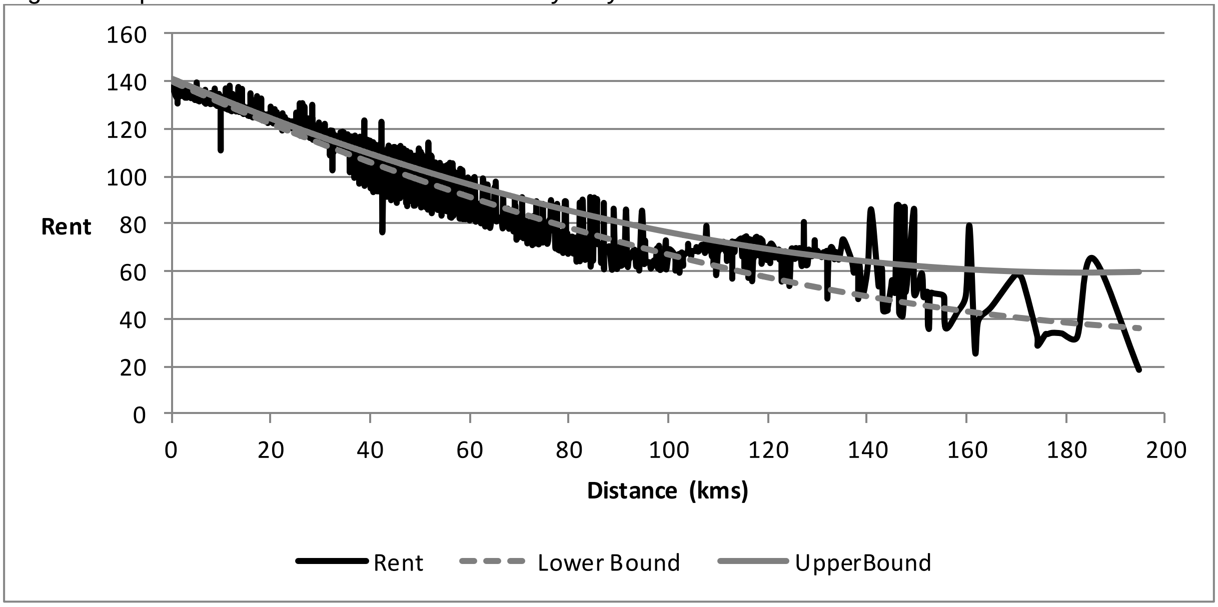

Table 1: Sydney GMA, Spatial concentrations (HH) of employment accessibility (μ), 2011

SA4 No	SA4 name	No of TZs
102	Central Coast	20
106	H. Valley exc Newcastle	16
111	Newcastle & L. Macquarie	7
114	S. Highlands & Shoalhaven	8
115	Baulkham Hills & Hawkesbury	2
117	City & Inner South	12
118	Eastern Suburbs	4
121	North Sydney & Hornsby	11
122	Northern Beaches	13
	Outer West & Blue Mountains	1
125	Parramatta	1
126	Ryde	1
128	Sutherland	1

Finally we computed the Moran's I statistic based on 9999 permutations to explore the spatial distributions of the exponents β and γ in order to cast some light on Fotheringham's (1983) claim about competing destinations. The Moran I for the origin based βs were -0.001 for 2011 which is insignificant at the 5% level, whereas the Moran I for the γs for 2011 were 0.245, which is significant at the 1% level, which confirms Fotheringham's contention.

Figure 2: Sydney GMA, HH and LL concentrations of employment accessibility, 2011

5. Conclusion

This exploratory study has been designed to investigate the application of a sophisticated spatial interaction model, incorporating more constraints than the standard DCSI, to commuting flows defined at a high level of spatial disaggregation in a large urban area. The results reveal that, despite the six negative impedance exponents, the underlying form of the spatial interaction model could be applied to Sydney GMA commuting flow data, notwithstanding the inevitable errors and inconsistencies of Census data and the data randomisation undertaken by the ABS. Further analysis of these six anomalous results is required

Following the work of O'Kelly et al. (2012), one novelty of the approach has been the derivation of implicit rents (and wages) by TZ, which has enabled analysis of the rent gradient. At this stage whether the rent gradient can generate any insights about polycentricity is unclear. There is not a linear relationship between rent and distance from the CBD. The results do not suggest that this is simply a product of polycentricity, but could also reflect the relatively simple assumptions made by Alonso (1964). Second, the appropriateness of the measure of destination accessibility as a means of identifying polycentric locations also needs further consideration. The presence of HH combinations of employment accessibility outside the metropolitan area does suggest a more nuanced interpretation of accessibility, as compared to arbitrary measures based on the magnitude of local employment. Finally the application of spatial statistics, such as the LISA and Getis-Ord, to identify concentrations of areas of accessibility also warrants further consideration.

Appendix

In matrix form, equations (11) and (13) can be written as:

$$\frac{d\lambda}{d\varphi} + T^d \frac{d\mu}{d\varphi} = \bar{C}_{*.} \qquad \text{(A1)}$$

and

$$\frac{d\mu}{d\varphi} + T^o \frac{d\lambda}{d\varphi} = \bar{C}_{.*} \qquad \text{(A2)}$$

where $\frac{d\lambda}{d\varphi}$ and $\frac{d\mu}{d\varphi}$ denote column vectors of derivatives, T^d denotes the conditional probability, $T_{j|i}$ of commuting to destination j given departure from origin i, T^o denotes the conditional probability, $T'_{i|j}$ of commuting from origin i given arrival at destination j and $\bar{C}_{*.}$, $\bar{C}_{.*}$ are respectively the column vectors of origin and destination based average commutes (O'Kelly, 2012). He shows that this set of simultaneous equations for the unknowns, $\frac{d\lambda}{d\varphi}$ and $\frac{d\mu}{d\varphi}$, can be solved to obtain the following:

$$\frac{d\lambda}{d\varphi} = (I - T^d T^o)^{-1}(\bar{C}_{*.} - T^d \bar{C}_{.*}) \qquad \text{(A3)}$$

$$\frac{d\mu}{d\varphi} = (I - T^o T^d)^{-1}(\bar{C}_{.*} - T^o \bar{C}_{*.}) \qquad \text{(A4)}$$

In the absence of a numeraire both matrices to be inverted are singular. The row of Td corresponding to the numeraire is set to zero, and also the corresponding elements of $\bar{C}_{*.}, \bar{C}_{.*}$.

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